MINERνA & NOvA

Technical Note

Calculate like a Llewellyn

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The steps required to calculate cross sections of interest to neutrino physicists are discussed in many textbooks. Acquiring facility with the relevant skill set, however, is usually left to self-study. The net result is that theorists learn how while most experimentalists do not. This is unfortunate because ν_ℓ cross sections in the sub- through multiple-GeV range of E_ν are 'of a kind'. In this range, current-current phenomenology with four-fermion effective coupling is completely adequate, and it is not hard to work with it.

This Note shows how the component pieces play out in a calculation that can readily be generalized to most any neutrino-nucleon semi-leptonic cross section involving scattering into a two-body final state. The designated target is the neutrino-nucleon quasi-elastic scattering formulas of Llewellyn Smith's 1972 review article on neutrino interactions [1]. While a number of other equivalent expressions of these formulas appeared in the literature before Llewellyn Smith's review [2, 3, 4], the Llewellyn Smith (LS) formulation is notably clear, compact, and free of errors, consequently it is the foundational work usually cited in experimental and phenomenological treatments of $\nu_\ell/\bar{\nu}_\ell$ quasielastic scattering.

The intent of this Note is purely pedagogical – there is nothing new in the derivation presented herein. Facility with cross-section calculations requires some knowledge of dirac-ology and familiarity with basic trace theorems; for convenience, prerequisite formulas of this kind are given in the Appendix. Clear explanations of various aspects of the methodology can be found in **Thompson** ([5]: Sec. 6.5.1- 6.5.3) and in **Halzen and Martin** ([6]: Ch. 6); **Quigg** [7] provides a number of illustrative calculations. A calculation that is more complete than the one presented here in that it includes exotic form factors, can be found in Appendix C of Teppei Katori's doctoral thesis [8]. However, for Neutrino Enthusiasts who 'wanna learn how to boogie' – read on !

1 Currents, form factors, and kinematics

We consider neutrino quasi-elastic scattering on a free neutron:

$$\nu_\ell(k_1) + \mathrm{n}(p_1) \rightarrow \ell^-(k_2) + p(p_2). \tag{1}$$

As Fig. 1 illustrates, the interaction involves the coupling of a leptonic current, $j^{(\ell)\mu}$, to a the hadronic current, $J^{(h)}_\mu$, via an effective four-fermion interaction whose coupling strength is $G_F \cos\theta_C/\sqrt{2}$. Four-momenta of the initial- and final state leptons (hadrons) are designated k_1, k_2 (p_1, p_2) respectively.

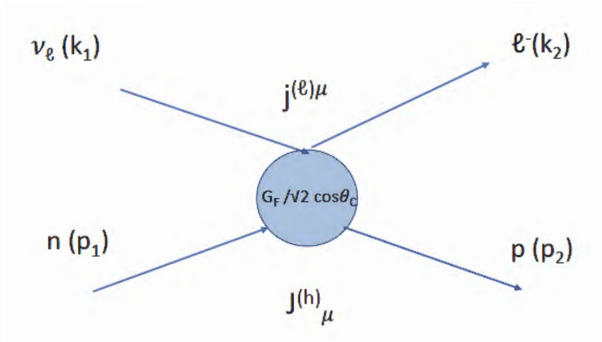

Figure 1: Momentum four-vectors for the initial and final-state particles of reaction (1).

The leptonic current is

$$j^{(\ell)\mu} = \overline{u}(\ell, k_2)\,\gamma^\mu(1-\gamma_5)\,u(\nu_\ell, k_1) = \overline{u}(k_2)\,\gamma^\mu(1-\gamma_5)\,u(k_1). \tag{2}$$

The hadronic current, after application of Gordon's decomposition which reduces the vector magnetic moment term (see Ref. [6], Sec. 6.1), can be written

$$J^{(h)}_\mu = \overline{u}(p_2)\,[\gamma_\mu\, a - P_\mu\, b + \gamma_\mu\gamma_5\, c]\,u(p_1) \tag{3}$$

where

$$P_\mu = (p_1+p_2)_\mu \;\text{ and }\; a = F_V^1(Q^2) + \xi F_V^2(Q^2), \quad b = F_V^2(Q^2)/2M, \quad c = F_A(Q^2). \tag{4}$$

Here, the functions a, b, and c provide shorthand expressions for terms involving the vector and axial vector form factors of Llewellyn Smith's notation. The nucleon mass $M = (M_p + M_n)/2$ is used throughout, and the proton-neutron mass difference is neglected.

We define some kinematic quantities at the outset, emphasizing variables which are Lorentz-invariants and/or are constructed using both particles of either of the currents. Useful variables are the four-momentum sums, the four-momentum transfer and related variables, and the Mandelstam variables s and u. Suppressing four-vector indices, these variables are defined as

$$P = p_1 + p_2\,, \;\; K = k_1 + k_2\,, \quad q = k_1 - k_2 = p_2 - p_1\,, \tag{5}$$

$$s = (p_1+k_1)^2 = (p_2+k_2)^2\,, \quad u = (k_2-p_1)^2 = (p_2-k_1)^2. \tag{6}$$

$$Q^2 = -q^2\,, \quad \tau \equiv -q^2/4M^2 = Q^2/4M^2\,, \tag{7}$$

In the LS paper, q^2 is used throughout. Since this variable is negative, many researchers prefer to use either Q^2 or the scaled four-momentum transfer squared, τ, defined in Eq. (7). In this Note, the cross section finally calculated will be expressed in terms of τ.

Kinematic relations involving the above variables are presented below. They can be readily deduced, either by working with the above definitions or by carrying out evaluations in the CM or the Lab frame. These relations will prove extremely helpful with reducing formulas that emerge when the the leptonic tensor is contracted with the hadronic tensor.

$$K\cdot P = s-u\,, \quad K\cdot K = -q^2 + 2\,m_\ell^2\,, \quad P\cdot P = 4M^2 - q^2\,, \quad P\cdot q = 0\,, \quad K\cdot q = -m_\ell^2\,, \tag{8}$$

and

$$p_1 \cdot p_2 = M^2 - \frac{1}{2}q^2\,, \qquad (s-u) = 4ME_\nu + q^2 - m_\ell^2\,. \tag{9}$$

The calculational methodology is best illustrated by minimizing 'clutter'. To that end it is useful to neglect the final-state lepton mass, m_ℓ. As it turns out, this does not degrade the calculation in a fundamental way since the forms of the neglected terms can be straightforwardly inferred from the experience gained from calculating without them. Terms proportional to powers of m_ℓ will be recovered at the end. Neglect of the lepton mass simplifies some of the relations of Eqs. (8) and (9), e.g.

$$K \cdot K \simeq -q^2\,, \quad K \cdot q \simeq 0\,, \quad (s-u) \simeq 4ME_\nu + q^2\,. \tag{10}$$

From the local property of the lepton current it follows that the transition probability for Eq. (1) must take the form of a polynomial expansion up to quadratic in the variable $(s-u)$ [3].

2 Assembly of the squared invariant amplitude

The centerpiece of the calculation is the determination of the square of the invariant matrix element for reaction (1),

$$|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^* = \mathcal{M}\mathcal{M}^\dagger = (j^{(\ell)\mu}J^{(h)}_\mu)(J^{(h)\dagger}_\nu j^{(\ell)\nu\dagger}) = [j^{(\ell)\mu}j^{(\ell)\nu\dagger}]\,[J^{(h)}_\mu J^{(h)\dagger}_\nu] = L^{\mu\nu}W_{\mu\nu}\,, \tag{11}$$

where expression of the overall interaction strength, $(G_F\cos\theta_C)^2/2$, is suppressed here and in all intermediate steps of the calculation.

It will be necessary to average over the target spin and to sum over the spins of the final-state particles. These steps will be subsumed in calculation of the leptonic and hadronic tensors, and their presence will be indicated using overbars, e.g.

$$|\mathcal{M}|^2 \to \text{sum over initial spins, average over final spins} \to \overline{|\mathcal{M}|^2} = \overline{L}^{\mu\nu}\overline{W}_{\mu\nu}\,. \tag{12}$$

2.1 Calculation of the leptonic tensor

As can be seen in Eq. (11), the leptonic tensor is the bilinear form of lepton currents

$$L^{\mu\nu} = j^{(\ell)\mu}j^{(\ell)\nu\dagger} = \{\overline{u}(k_2)\,[\gamma^\nu(1-\gamma_5)]\,u(k_1)\}\;\{\overline{u}(k_1)\,[\gamma^\mu(1-\gamma_5)]\,u(k_2)\}\,. \tag{13}$$

This expression involves matrix multiplications and so some rearrangement is allowed (see Ref. [6], p. 122):

$$L^{\mu\nu} = \gamma^\mu(1-\gamma_5)\,u(k_1)\overline{u}(k_1)\,(1+\gamma_5)\gamma^\nu\,u(k_2)\overline{u}(k_2) \tag{14}$$

Here, $\overline{u}(k_2)$ is moved (cyclically, as allowed in a trace) and γ^ν has been moved to the right, generating a sign change as it moves by γ_5. Introducing summation over the final-state spins allows completeness relations for Dirac projector forms to be identified,

$$\sum_{s_2} u(k_2)\overline{u}(k_2) = \not{k}_2 + m_\ell \simeq \not{k}_2\,, \quad \text{and } u(k_1)\overline{u}(k_1) = \not{k}_1. \tag{15}$$

The sum over final spins and averaging over the initial spins allows the matrix multiplications to be identified with a trace; it also introduces the factor one-over-number-of-initial-spin-states, which is 1 for this current. It follows that

$$\overline{L}^{\mu\nu} = \mathbf{tr}\{\gamma^\mu(1-\gamma_5)\not{k}_1(1+\gamma_5)\gamma^\nu\not{k}_2\} \ . \tag{16}$$

The $(1+\gamma_5)$ can be moved to the left, giving $(1-\gamma_5)(1-\gamma_5) = 1 - 2\gamma_5 + \gamma_5^2 = 2(1-\gamma_5)$, hence

$$\overline{L}^{\mu\nu} = 2\,\mathbf{tr}\{\gamma^\mu(1-\gamma_5)\not{k}_1\gamma^\nu\not{k}_2\} = 2\,\mathbf{tr}\{(1+\gamma_5)\gamma^\mu\not{k}_1\gamma^\nu\not{k}_2\} = 2\,\mathbf{tr}\{\gamma^\mu\not{k}_1\gamma^\nu\not{k}_2\} + 2\,\mathbf{tr}\{\gamma_5\gamma^\mu\not{k}_1\gamma^\nu\not{k}_2\} \ . \tag{17}$$

The latter two terms are covered by basic trace theorems (Appendix: relations (88), (90)) and reduce to

$$\overline{L}^{\mu\nu} = 2\,\{4\,(k_1^\mu k_2^\nu - g^{\mu\nu}(k_1\cdot k_2) + k_1^\nu k_2^\mu) - 4i\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma}\} \ . \tag{18}$$

Equation (18) can be separated into terms that are symmetric (S) or antisymmetric (A) under exchange of four-vector indices:

$$\overline{L}^{\mu\nu}(S) = 8\,\{k_1^\mu k_2^\nu + k_1^\nu k_2^\mu + \frac{q^2}{2}g^{\mu\nu}\} \qquad \text{and} \qquad \overline{L}^{\mu\nu}(A) = -8i\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma} \ . \tag{19}$$

These forms are given in Ref. [9]. It is useful however to re-express them using our auxiliary four-vectors. We write

$$k_1^\mu k_2^\nu + k_1^\nu k_2^\mu = \frac{1}{2}(K^\mu K^\nu - q^\mu q^\nu) \ . \tag{20}$$

Additionally, we break $\overline{L}^{\mu\nu}(A)$ into two equal pieces and rearrange the antisymmetric tensor:

$$-8\,i\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma} = -4i\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma} - 4i\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma} \tag{21}$$

$$= -4i\epsilon^{\mu\nu\rho\sigma}k_{1\rho}k_{2\sigma} + 4i\epsilon^{\mu\nu\rho\sigma}k_{2\rho}k_{1\sigma} \tag{22}$$

$$= +4\,i\epsilon^{\mu\nu\rho\sigma}(k_{1\rho}k_{1\sigma} - k_{1\rho}k_{2\sigma} + k_{2\rho}k_{1\sigma} - k_{2\rho}k_{2\sigma}) = +4\,i\epsilon^{\mu\nu\rho\sigma}K_\rho\,q_\sigma \ . \tag{23}$$

Our final expression is

$$\overline{L}^{\mu\nu} = \overline{L}^{\mu\nu}(S) + \overline{L}^{\mu\nu}(A) = 4\,\{K^\mu K^\nu - q^\mu q^\nu + q^2 g^{\mu\nu}\} + 4\,i\epsilon^{\mu\nu\rho\sigma}K_\rho\,q_\sigma \ . \tag{24}$$

If the final-state lepton mass is retained rather than neglected, then $\overline{L}^{\mu\nu}(S)$ acquires an extra term: $-4\,m_\ell^2\,g^{\mu\nu}$.

2.2 Calculation of the hadronic tensor

To construct the hadronic tensor $\overline{W}_{\mu\nu}$, recall that

$$W_{\mu\nu} = J_\mu^{(h)}J_\nu^{(h)\dagger} = \overline{u}(p_2)\Gamma_\mu u(p_1)\cdot\overline{u}(p_1)\overline{\Gamma}_\nu u(p_2) \tag{25}$$

where

$$\Gamma_\mu = \overline{\Gamma}_\mu = [\,\gamma_\mu\,a\, -\, P_\mu\,b\, +\, \gamma_\mu\gamma_5\,c\,] \ . \tag{26}$$

Upon averaging over initial spins (which gives a factor of $\frac{1}{2}$) and summing over final spins, Eq. (25) becomes

$$\overline{W}_{\mu\nu} = \frac{1}{2}\,\mathbf{tr}\{\Gamma_\mu(\not{p}_1 + M)\Gamma_\nu(\not{p}_2 + M)\} \tag{27}$$

$$= \frac{1}{2}\,\mathbf{tr}\{\Gamma_\mu\, \not{p}_1 \Gamma_\nu\, \not{p}_2 + M\,\Gamma_\mu\, \not{p}_1 \Gamma_\nu\; + M\,\Gamma_\mu\,\Gamma_\nu\, \not{p}_2 + M^2\,\Gamma_\mu\,\Gamma_\nu\} \tag{28}$$

$$= \frac{1}{2}\,\mathbf{tr}\{\text{term 1} \quad + \quad \text{term 2} \quad + \quad \text{term 3} \quad + \quad \text{term 4}\}\ . \tag{29}$$

Trace evaluations for each of the four terms can be launched from a full expression of $\Gamma_\mu\,\Gamma_\nu$:

$$\begin{aligned}\Gamma_\mu\,\Gamma_\nu = \gamma_\mu\gamma_\nu\, a \;-\; \gamma_\mu P_\nu\, a\, b \;+\; \gamma_\mu\gamma_\nu\gamma_5\, a\, c \\ -\; P_\mu\gamma_\nu\, a\, b \;+\; P_\mu P_\nu\, b^2 \;-\; P_\mu\gamma_\nu\gamma_5\, b\, c \\ +\; \gamma_\mu\gamma_5\gamma_\nu\, a\, c \;-\; \gamma_\mu\gamma_5 P_\nu\, b\, c \;+\; \gamma_\mu\gamma_5\gamma_\nu\gamma_5\, c^2\ \ .\end{aligned} \tag{30}$$

The trace of Term 4 in Eq. (28) can be immediately evaluated using Eq. (30) because the terms involving ab, ac, and bc are all zero according to trace theorems (Appendix: (86), (89)). The remaining terms (using Appendix relations (85), (87)) give

$$\overline{W}_{\mu\nu}(4) = \frac{1}{2}\,M^2\,[4\,g_{\mu\nu}(a^2 - c^2) + 4\,P_\mu P_\nu\, b^2]\ . \tag{31}$$

Term 3 of Eq. (28) involves appending a $\not{p}_2$ onto each term of Eq. (30). The resulting non-zero terms are

$$\overline{W}_{\mu\nu}(3) = \frac{1}{2}\,M\,\mathbf{tr}\{-\gamma_\mu\, \not{p}_2 P_\nu\, ab - P_\mu\gamma_\nu\, \not{p}_2\, ab\} \tag{32}$$

$$= -\frac{1}{2}\,M\,ab\,(\,\mathbf{tr}\{\gamma_\mu\gamma_\lambda\}\,p_2^\lambda P_\nu + \mathbf{tr}\{\gamma_\nu\gamma_\lambda\}\,p_2^\lambda P_\mu\,)\ . \tag{33}$$

$$\overline{W}_{\mu\nu}(3) = -2M\,ab\,(\,p_{2\mu}P_\nu + p_{2\nu}P_\mu\,)\ . \tag{34}$$

Term 2 of Eq. (28) involves insertion of $\not{p}_1$ between Γ_μ and Γ_ν in each term of Eq. (30), and it reduces very similarly to Term 3:

$$\overline{W}_{\mu\nu}(2) = \frac{1}{2}\,M\,\mathbf{tr}\{-\gamma_\mu\, \not{p}_1 P_\nu\, ab - P_\mu \not{p}_1 \gamma_\nu\;\, ab\} = -2M\,ab\,(\,p_{1\mu}P_\nu + p_{1\nu}P_\mu\,)\ . \tag{35}$$

Equation (35) can be straightaway combined with Eq. (34) to obtain

$$\overline{W}_{\mu\nu}(2+3) = -4\,ab\,M(\,P_\mu P_\nu\,)\ . \tag{36}$$

Turning finally to Term 1 of Eq. (28), one sees that it involves insertion of both $\not{p}_1$ and $\not{p}_2$ into each term of Eq. (30). Once again, the trace theorems enable immediate elimination of some terms as they contain odd numbers of γ's, or they contain a γ_5 together with 1 or 2 or 3 γ's. The remaining terms give

$$\overline{W}_{\mu\nu}(1) = \frac{1}{2}\,\mathbf{tr}\{\gamma_\mu\, \not{p}_1 \gamma_\nu\, \not{p}_2\, a^2 + \gamma_\mu\, \not{p}_1 \gamma_\nu\gamma_5\, \not{p}_2\, ac + P_\mu\, \not{p}_1 P_\nu\, \not{p}_2\, b^2 + \gamma_\mu\gamma_5\, \not{p}_1 \gamma_\nu\, \not{p}_2\, ac + \gamma_\mu\gamma_5\, \not{p}_1 \gamma_\nu\gamma_5\, \not{p}_2\, c^2\}\ , \tag{37}$$

$$= \frac{1}{2}\,\mathbf{tr}\{\gamma_\mu\gamma_\lambda\gamma_\nu\gamma_\delta\, p_1^\lambda p_2^\delta\, a^2 - 2\,\gamma_5\gamma_\mu\gamma_\lambda\gamma_\nu\gamma_\delta\, p_1^\lambda p_2^\delta\, ac + \gamma_\lambda\gamma_\delta\, p_1^\lambda p_2^\delta P_\mu P_\nu\, b^2 + \gamma_\mu\gamma_\lambda\gamma_\nu\gamma_\delta\, p_1^\lambda p_2^\delta\, c^2\}\ , \tag{38}$$

$$= \frac{1}{2}\,\{(\,a^2 + c^2\,)\,4\,(g_{\mu\lambda}\,g_{\nu\delta} - g_{\mu\nu}\,g_{\lambda\delta}\, + g_{\mu\delta}\,g_{\lambda\nu})\,p_1^\lambda p_2^\delta + 4\,g_{\lambda\delta}\,p_1^\lambda p_2^\delta\,P_\mu P_\nu\, b^2 - 2\cdot 4\,i\epsilon_{\mu\lambda\nu\delta}\,p_1^\lambda p_2^\delta\, ac\}\ . \tag{39}$$

Our final expression for $\overline{W}^{\mu\nu}(1)$ is

$$\overline{W}^{\mu\nu}(1) = \frac{1}{2}\,\{(\,a^2 + c^2\,)\,4\,(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu}(p_1\cdot p_2)) + 4\,(p_1\cdot p_2)P_\mu P_\nu\, b^2\; + 8\,i\epsilon_{\mu\nu\lambda\delta}\,p_1^\lambda p_2^\delta\, ac\}\ . \tag{40}$$

Note that Eq. (40) has terms whose structure resembles the terms of $\overline{L}^{\mu\nu}$ in Eq. (18). As a consequence we can utilize a kinematic relation that has resemblance to Eq. (20):

$$p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} = \frac{1}{2}(P_\mu P_\nu - q_\mu q_\nu) \ . \tag{41}$$

Using Eq. (41) with Eq. (9), we obtain

$$\overline{W}^{\mu\nu}(1) = \frac{1}{2}\{(a^2+c^2)\,(2(P_\mu P_\nu - q_\mu q_\nu) - (4M^2-2q^2)g_{\mu\nu}) + (4M^2-2q^2)P_\mu P_\nu\, b^2 \ + 8\, i\epsilon_{\mu\nu\lambda\delta}\, p_1^\lambda p_2^\delta\, ac\} \tag{42}$$

$$= 2\,\{(a^2+c^2)\,(\frac{1}{2}(P_\mu P_\nu - q_\mu q_\nu) - (M^2 - \frac{q^2}{2})g_{\mu\nu}) + (M^2 - \frac{q^2}{2})P_\mu P_\nu\, b^2 \ + 2\, i\epsilon_{\mu\nu\lambda\delta}\, p_1^\lambda p_2^\delta\, ac\} \ . \tag{43}$$

The last term can be rearranged following the steps shown previously that lead to Eq. (23). Our final result is

$$\overline{W}^{\mu\nu}(1) = 2\{(a^2+c^2)[\frac{1}{2}(P_\mu P_\nu - q_\mu q_\nu) - (M^2 - \frac{q^2}{2})g_{\mu\nu}] + (M^2 - \frac{q^2}{2})P_\mu P_\nu\, b^2 \ - ac\, i\epsilon_{\mu\nu\lambda\delta}\, q^\lambda P^\delta\} \ . \tag{44}$$

Gathering together Eqs. (31), (36), and (44), and noting that the only term in $\overline{W}_{\mu\nu}$ that is antisymmetric under exchange of four-vector indices is the final term in (44), we write $\overline{W}_{\mu\nu} = \overline{W}_{\mu\nu}(S) + \overline{W}_{\mu\nu}(A)$, where

$$\overline{W}^{\mu\nu}(S) = \frac{1}{2}M^2[4(a^2-c^2)g_{\mu\nu} \quad + \quad 4b^2 P_\mu P_\nu] - 4Mab\, P_\mu P_\nu$$

$$+ \ 2(a^2+c^2)[\frac{1}{2}(P_\mu P_\nu - q_\mu q_\nu) - (M^2 - \frac{q^2}{2})g_{\mu\nu}] + 2\, b^2\, (M^2 - \frac{q^2}{2})P_\mu P_\nu \ , \tag{45}$$

and

$$\overline{W}^{\mu\nu}(A) = -\,2\, i\, ac\, \epsilon_{\mu\nu\lambda\delta}\, q^\lambda P^\delta\} \ . \tag{46}$$

From inspection of Eq. (45), it can be seen that the terms of $\overline{W}_{\mu\nu}(S)$ can be arranged to follow the ordering of terms in $\overline{L}_{\mu\nu}(S)$:

$$\overline{W}^{\mu\nu}(S) = [2\, M^2 b^2 - 4\, M\, ab + (a^2+c^2) + 2\, b^2(M^2 - \frac{q^2}{2})]P_\mu P_\nu \tag{47}$$

$$- \ (a^2+c^2)q_\mu q_\nu \ + \ [4\,\frac{M^2}{2}\,(a^2-c^2) - 2(a^2+c^2)(M^2 - \frac{q^2}{2})]g_{\mu\nu} \ . \tag{48}$$

We designate the coefficients of the quantities that carry the 4-vector indices as functions X, Y, and Z:

$$X = [2\, M^2 b^2 - 4\, M\, ab + (a^2+c^2) + 2\, b^2(M^2 - \frac{q^2}{2})] \ , \tag{49}$$

$$Y = -\ (a^2+c^2) \ , \tag{50}$$

$$Z = [4\,\frac{M^2}{2}\,(a^2-c^2) - 2(a^2+c^2)(M^2 - \frac{q^2}{2})] \ . \tag{51}$$

With these functions, $\overline{W}^{\mu\nu}(S)$ can be written compactly as

$$\overline{W}^{\mu\nu}(S) = X \cdot P_\mu P_\nu + Y \cdot q_\mu q_\nu + Z \cdot g_{\mu\nu} \ . \tag{52}$$

3 Contraction of $\overline{L}^{\mu\nu} \cdot \overline{W}_{\mu\nu}$

We are now in position to calculate $\overline{|\mathcal{M}|^2}$ as $\overline{L}^{\mu\nu}(S)\,\overline{W}_{\mu\nu}(S) \; + \overline{L}^{\mu\nu}(A)\;\overline{W}_{\mu\nu}(A)$, where

$$\overline{L}^{\mu\nu} = 4\,\{K^\mu K^\nu - q^\mu q^\nu + q^2 g^{\mu\nu}\} + 4\,i\epsilon^{\mu\nu\rho\sigma} K_\rho\, q_\sigma \qquad \text{and} \tag{53}$$

$$\overline{W}^{\mu\nu} = X \cdot P_\mu P_\nu + Y \cdot q_\mu q_\nu + Z \cdot g_{\mu\nu} \; - 2\,i\,ac\,\epsilon_{\mu\nu\lambda\delta} q^\lambda P^\delta \ . \tag{54}$$

We first carry out the contraction of the antisymmetric terms:

$$\overline{L}^{\mu\nu}(A)\;\overline{W}_{\mu\nu}(A) = (4\,i\epsilon^{\mu\nu\rho\sigma} K_\rho\, q_\sigma)(-2\,i\,ac\epsilon_{\mu\nu\lambda\delta} q^\lambda P^\delta) \tag{55}$$

$$= 4 \cdot 2 \cdot ac\,(\,-2\,[\delta^\rho_\lambda\,\delta^\sigma_\delta - \delta^\rho_\delta \delta^\sigma_\lambda]\,) K_\rho\, q_\sigma\, q^\lambda\, P^\delta \ , \tag{56}$$

$$= -16 \cdot ac\,[K_\lambda\, q_\delta\, q^\lambda\, P^\delta - K_\delta\, q_\lambda\, q^\lambda\, P^\delta \ , \tag{57}$$

$$= -16 \cdot ac\,[\,(K \cdot q)(q \cdot P) - (P \cdot K)\, q^2] \ . \tag{58}$$

Recall that $q \cdot P = 0$ and also $K \cdot q \simeq 0$, consequently

$$\overline{L}^{\mu\nu}(A)\;\overline{W}_{\mu\nu}(A) = +16\,ac\,(P \cdot K)\, q^2 = +16\,ac\, q^2\,(s-u) \ . \tag{59}$$

Expression of this result in LS notation gives

$$\overline{L}^{\mu\nu}(A)\;\overline{W}_{\mu\nu}(A) = (-)\,16\,M^2\,(\frac{-q^2}{M^2})\,F_A\,(F_V^1 + \xi F_V^2) = -16\,M^2\,B(q^2) \ , \tag{60}$$

where $B(q^2)$ is the functional form associated with (s - u) in the LS differential cross section.

We now proceed to the contract the symmetric terms, $\overline{L}^{\mu\nu}(S)\,\overline{W}_{\mu\nu}(S)$:

$$\begin{aligned} &= 4\,X\,(K \cdot P)^2 + 4\,Y\,(K \cdot q)^2 + 4\,Z\,(K \cdot K) \\ &\quad - 4\,X\,(P \cdot q)^2 - 4\,Y\,q^4 - 4\,Z\,q^2 \\ &\quad + 4\,X\,q^2\,(P \cdot P) + 4\,Y\,q^4 + 16\,Z\,q^2 \ . \end{aligned} \tag{61}$$

We recognize that $P \cdot q = 0$ and that $K \cdot q \simeq 0$ and that the two terms $\pm\,4\,Y\,q^2$ are self-canceling. Additionally, $K \cdot P = (s-u)$, $K \cdot K \simeq -q^2$, and $P \cdot P = 4\,M^2 - q^2$. Thus Eq. (61) reduces to

$$\overline{L}^{\mu\nu}(S)\,\overline{W}_{\mu\nu}(S) = 4\,X\,(s-u)^2 + 8\,Z\,q^2 + 4\,X\,q^2\,(4\,M^2 - q^2) \ . \tag{62}$$

Let us consider the first term in Eq. (62); we put aside the factor $(s-u)^2$ and evaluate $4\,X$:

$$4\,X = 4\,[\,c^2 + a^2 - 4M\,ab + 2M^2\,b^2 + 2b^2\,(M^2 - \frac{q^2}{2}\,)] \tag{63}$$

$$= 4\,[\,c^2 + a^2 - 4M\,ab + 4M^2\,b^2 - q^2\,b^2] \tag{64}$$

$$= 4\,[\;|F_A|^2 + |F_V^1|^2 + 2\,F_V^1\,\xi F_V^2 + |\xi F_V^2|^2 - 4\,M(F_V^1 + \xi F_V^2\,)\frac{\xi F_V^2}{2M} + 4M^2\frac{|\xi F_V^2|^2}{4M^2} - q^2\,\frac{|\xi F_V^2|^2}{4M^2}\;] \ . \tag{65}$$

This reduces to

$$4\,X\,(s-u)^2 = 4\,[\;|F_A|^2 + |F_V^1|^2 - \frac{q^2}{4M^2}\,|\xi F_V^2|^2\;](s-u)^2 \ . \tag{66}$$

This term corresponds to $16 \cdot C(q^2)$ in the LS expression for the quasielastic differential cross section. Thus, with terms (60) and (66), we have accounted for LS terms that are proportional to $(s-u)$ and $(s-u)^2$. This means that the remaining terms in Eq. (62) must be proportional to the LS form $A(q^2)$. These remaining terms - designated as $\mathcal{R}$ - are

$$\mathcal{R} \,=\, +8\,Z\,q^2 \,+\, 4\,X\,q^2\,(\,4M^2 - q^2\,) = -\,8\,Z\,(-q^2) \,-\, 4\,X\,(-q^2)\,4M^2(\,1 - \frac{q^2}{4M^2}\,)\,. \tag{67}$$

This can be expressed more compactly using the scaled four-momentum transfer, τ:

$$\mathcal{R} \,=\, -\,8\,Z\,4M^2\,\tau \,-\, 4\,X\,(4M^2)^2\,4M^2\,\tau\,(\,1+\tau\,) = 8\cdot 4M^2\,\tau[\,-Z \,-\, 2M^2\,X\,(1+\tau)\,]\,. \tag{68}$$

From Eq. (66), X can be expressed in terms of form factors in LS notation. Similarly, Z can be rewritten as follows:

$$Z \,=\, 2M^2[\,a^2 - c^2 - (a^2+c^2)(1+2\tau)\,] = 2M^2\,[\,-2\,c^2\,(1+\tau) \,-\, 2\,a^2\,\tau\,]. \tag{69}$$

In LS notation, this becomes

$$Z \,=\, 2M^2[\,-2\,|F_A|^2\,(1+\tau) - 2\,|F_V^1|^2\,\tau - \,4F_V^1\,\xi F_V^2\,\tau - \,2\,|\xi F_V^2|^2\,\tau\,]. \tag{70}$$

We now combine Eq. (70) for Z with our knowledge of X from Eq. (66), and write the remaining terms $\mathcal{R}$ as

$$\mathcal{R} = \,8\,(4M^2)\,(2M^2)\,\tau\,[\,2\,|F_A|^2\,(1+\tau) + 2\,|F_V^1|^2\,\tau + \,4F_V^1\,\xi F_V^2\,\tau + \,2\,|\xi F_V^2|^2\,\tau \tag{71}$$

$$-\,|F_A|^2\,(1+\tau) - |F_V^1|^2\,(1+\tau) - |\xi F_V^2|^2\,\tau(1+\tau)\,]\,, \tag{72}$$

which reduces to

$$\mathcal{R} = \,64\,M^4\,\tau\,[\,|F_A|^2\,(1+\tau) - \,|F_V^1|^2\,(1-\tau) \,+\, |\xi F_V^2|^2\,\tau\,(1-\tau) + \,4F_V^1\,\xi F_V^2\,\tau\,]\,. \tag{73}$$

This expression is related to Llewellyn's form $A(q^2)$ as follows: $\mathcal{R} = (4\,M^2)^2 \cdot A(q^2)$.

In summary, the result of contracting together the leptonic and hadronic tensors is

$$\overline{L}^{\mu\nu}\,\overline{W}_{\mu\nu} = (4\,M^2)^2 \cdot A(q^2) \,-\, \frac{(s-u)}{M^2}\,(4\,M^2)^2 \cdot B(q^2) + \,\frac{(s-u)^2}{M^4}\,(4\,M^2)^2 \cdot C(q^2)\,. \tag{74}$$

4 The differential cross section

The general form for the differential cross section in the LAB frame (target nucleon at rest) is

$$\frac{d\sigma}{dQ^2} = \frac{1}{64\pi\,E_\nu^2 M^2}\,\frac{G_F^2\cos^2\theta_C}{2}\,\overline{|\mathcal{M}|^2}\,, \tag{75}$$

$$= \,\frac{G_F^2\cos^2\theta_C}{2}\,\frac{1}{64\pi\,E_\nu^2 M^2}\,(4M^2)^2\,\frac{\overline{L}^{\mu\nu}\,\overline{W}_{\mu\nu}}{(4M^2)^2}\,. \tag{76}$$

Thus we arrive at the Llewellyn Smith differential cross section result as expressed in LS (3.18) and (3.22), for the case where the final-state lepton mass is neglected:

$$\frac{d\sigma}{dQ^2}(\nu_\mu n \to \mu^- p) = \frac{G_F^2\cos^2\theta_C M^2}{8\pi\,E_\nu^2}\,\,[A(\tau) - B(\tau)\,\frac{(s-u)}{M^2} + C(\tau)\,\frac{(s-u)^2}{M^4}\,]\,, \tag{77}$$

where

$$A(\tau) = 4\tau\,[\ |F_A|^2\,(1+\tau) - \ |F_V^1|^2\,(1-\tau)\ +\ |\xi F_V^2|^2\,\tau\,(1-\tau) + \ 4\,F_V^1\,\xi F_V^2\,\tau\ ]\ , \tag{78}$$

$$B(\tau) = (-)4\tau\,[\,F_A\,(F_V^1 + \xi F_V^2)\,]\ , \tag{79}$$

$$C(\tau) = \frac{1}{4}\,[\ |F_A|^2 + |F_V^1|^2 + \tau\,|\xi F_V^2|^2\ ]\ . \tag{80}$$

For reasons previously stated, the cross section is here expressed in terms of the scaled four-momentum transfer squared, $\tau = \frac{-q^2}{4\,M^2}$, instead of $-q^2 = Q^2$: Inclusion of terms proportional to the lepton mass follows straightforwardly from our derivation, as described below.

5 Four features of the quasielastic cross section

1) Our calculation neglects the final-state lepton mass, however we can identify where and how its inclusion would modify Eqs. (78), (79), and (80):

- As remarked after Eq. (24), retention of m_ℓ yields an extra term in $\overline{L^{\mu\nu}}$, namely $-4\,m_\ell^2\,g_{\mu\nu}$.
- Certain kinematic factors that enter when $\overline{L^{\mu\nu}}$ is contracted with $\overline{W_{\mu\nu}}$ must be specified more precisely. These factors are $K \cdot q = -m_\ell^2$ and $K \cdot K = 2m_\ell^2 - q^2$.
- The result of these modifications is that the $A(\tau)$ form gets expanded, however the $B(\tau)$ and $C(\tau)$ forms remain the same.
- The approximate form $A(\tau)$ given by Eq. (78) is modified as follows:

$$A \to A' = A + \frac{m_\ell^2}{M^2}\,A - 4\,(\frac{m_\ell^2}{M^2})(\frac{m_\ell^2}{4M^2} + \tau)\,(\,|F_A|^2 + |F_V^1 + \xi F_V^2|^2\,). \tag{81}$$

- For quasielastic scattering by ν_e and ν_μ neutrinos, these terms have little impact. For scattering of ν_τ neutrinos however, they are important. They enable the pseudoscalar form factor, $F_P(Q^2)$, neglected here, to play a major role.

2) For quasielastic scattering initiated by antineutrinos,

$$\overline{\nu}_\ell(k_1) + \mathrm{p}(p_1) \to \ell^+(k_2) + n(p_2)\ , \tag{82}$$

the calculation is nearly identical and the only change to the final result, Eq. (77), is that the $B(\tau)$ term changes sign.

3) In the limit $Q^2 \to 0$ (hence $\tau \to 0$), it can be seen from Eqs. (78), (79), and (80) that $A \to 0$, $B \to 0$, and $C \to \frac{1}{4}\,(|F_A(0)|^2 + |F_V^1(0)|^2) \simeq 0.63$. Additionally $(s-u)^2 \to (4M)^2\,E_\nu^2$, consequently

$$\frac{d\sigma}{dQ^2}|_{Q^2=0} = \frac{G_F^2\cos^2\theta_C}{2\pi}\,(|F_A(0)|^2 + |F_V^1(0)|^2)\ , \tag{83}$$

which is independent of E_ν !

4) At very high energies, $s \to \infty$, the $C(\tau)$ term dominates the cross section.

Appendix: Trace theorems and useful relations

For evaluating the matrix element squared, the following relation holds for all Standard Model interactions (Ref. [5], Sec. 6.5.2),

$$[\overline{\psi}\,\Gamma\,\phi]^{\dagger} = [\overline{\phi}\,\Gamma\,\psi]\ , \tag{84}$$

where Γ is an interaction vertex form, e.g. γ^{μ} or $\gamma^{\mu}(1-\gamma^{5})$, etc.

It is useful to know that

$$g_{\mu\nu}g^{\mu\nu} = 4\ ,\quad \gamma^{5}\gamma^{5} = \mathrm{I}\ ,\quad \{\gamma^{5},\gamma^{\mu}\}_{+} = 0\ ,\quad tr\{\mathrm{I}\} = 4,\quad tr\{ABC\} = tr\{BCA\}\ . \tag{85}$$

With evaluating traces of Feynman-slashed four-vectors, recognize that the four-vector can be "unloaded" from the trace, e.g. $\mathbf{tr}\{\gamma^{\mu}\not{p}\} = \mathbf{tr}\{\gamma^{\mu}\gamma^{\sigma}\}p_{\sigma}$.

Here are the basic trace theorems. Proofs for most can be found in Refs. [5] (Sec. 6.5.3) and [10] (Sec. 7.2).

$$\mathbf{tr}\{\gamma_{\mu}\} = 0\ ,\ \ \text{and moreover}\ \ \mathbf{tr}\{\text{odd number of Dirac } \gamma \text{ matrices}\} = 0\ . \tag{86}$$

$$\mathbf{tr}\{\gamma_{\mu}\gamma_{\nu}\} = 4\,g_{\mu\nu}\quad \text{which implies}\quad \mathbf{tr}\{\not{p}\not{q}\} = 4\,p\cdot q\ . \tag{87}$$

$$\mathbf{tr}\{\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\gamma_{\sigma}\} = 4\,[g_{\mu\nu}\,g_{\rho\sigma} - g_{\mu\rho}\,g_{\nu\sigma} + g_{\mu\sigma}\,g_{\nu\rho}]\ . \tag{88}$$

$$\mathbf{tr}\{\gamma_{5}\} = 0\ ,\ \ \mathbf{tr}\{\gamma_{5}\gamma_{\mu}\} = 0\ ,\ \ \mathbf{tr}\{\gamma_{5}\gamma_{\mu}\gamma_{\nu}\} = 0\ ,\ \ \mathbf{tr}\{\gamma_{5}\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\} = 0\ . \tag{89}$$

$$\mathbf{tr}\{\gamma_{5}\gamma_{\mu}\gamma_{\nu}\gamma_{\rho}\gamma_{\sigma}\} = 4\,i\,\epsilon_{\mu\nu\rho\sigma}\ . \tag{90}$$

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