MINERνA & NOvA

Technical Note

Isospin relations among CC(π) channels

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This Note resurrects some isospin relations involving I = 3/2 and I = 1/2 amplitudes that shed light on relative strengths among cross sections for neutrino production of single pions in scattering from quasi-free nucleons. Statements of these relations appeared in presentations of bubble chamber measurements of neutrino scattering during the 1970's and 80's – invariably without a demonstration of their origin – see for example Ref. [1]. These relations were featured in lectures given by L. M. Sehgal [3] and they appear as Eqs. (2.24) in Rein and Sehgal's important paper of 1981: *Neutrino-Excitation of Baryon Resonances and Single Pion Production* [2] – but again, without elaboration. As will be shown in paragraphs below, the relations follow straightforwardly from pairwise utilizations of Clebsch-Gordon coefficients. The relations enable a number of quantitative statements to be made about neutrino cross sections for CC single pion production; these are developed and briefly discussed. These inferences may be of interest today as guidelines for relative cross-section strengths to be expected for neutrino-hydrocarbon scattering in the few-GeV region of incident E_ν.

1 Neutrino-induced charged-current single pion reactions

Consider inelastic charged current (CC) neutrino scattering on quasi-free nucleons where the final hadronic state consists of a nucleon accompanied by a single pion:

$$\nu_\mu + \mathrm{p} \rightarrow \mu^- + p + \pi^+, \tag{1}$$

$$\nu_\mu + \mathrm{n} \rightarrow \mu^- + p + \pi^0, \tag{2}$$

$$\nu_\mu + \mathrm{n} \rightarrow \mu^- + n + \pi^+. \tag{3}$$

In what follows we focus on isospin relationships among the above three channels. For antineutrino CC(π) production there is a corresponding triplet of channels for which the isospin constraints are in one-to-one correspondence with those developed for the neutrino channels. The antineutrino reactions are:

$$\bar{\nu}_\mu + \mathrm{n} \rightarrow \mu^+ + n + \pi^-, \tag{4}$$

$$\bar{\nu}_\mu + \mathrm{p} \rightarrow \mu^+ + n + \pi^0, \tag{5}$$

$$\bar{\nu}_\mu + \mathrm{p} \rightarrow \mu^+ + p + \pi^-. \quad (6)$$

Here, the isospin content of reactions (4) versus (1), (5) versus (2), and (6) versus (3) is related by the mapping $|I, I_z\rangle \rightarrow |I, -I_z\rangle$. As a result isospin relations among the antineutrino channels can be obtained from relations for the neutrino channels by simultaneously swapping (4) for (1), (5) for (2), and (6) for (3).

2 Effective isospin amplitudes

Expressed in terms of the current x current phenomenology of yesteryear, the hadronic current in reactions (1) through (3) is driven by the $\Delta S = 0$ charged current operator J^+_μ which transforms as an isovector. The operator acts on a proton or neutron to give an isospin state $|+1, \pm 1/2\rangle$ which can be decomposed via C-G coefficients into $|3/2, +1/2\rangle \equiv A_3$ and $|1/2, +1/2\rangle \equiv A_1$, and these couple via C-G coefficients to a specific final state $\langle \pi, N|$. In the modern view, the W^+ gauge boson carries isovector charge to the hadronic vertex, and with the initial nucleon, gives an isospin state $|W^+ N\rangle = |+1, \pm 1/2\rangle$ which can be decomposed into A_3 and A_1, and these couple to a specific final state $\langle \pi, N|$ via C-G coefficients.

In a full calculation there arises a consideration that the charged current $J_1 \pm iJ_2$ "does not have the normalization for the Clebsch-Gordon coefficients, it must be normalized as $(J_1 \pm iJ_2)/\sqrt{2}$ which brings an additional factor of $\sqrt{2}$ to each of the charged current in comparison with the Clebsch-Gordon coefficients" [4]. In the derivations below this factor of $\sqrt{2}$ is included explicitly in order to reproduce the relations stated as Eqs. (2.24) in Rein and Sehgal [2]. In the end however this overall factor has no effect on the relations among amplitudes and cross sections that ensue, and so it can be omitted without consequence [1].

Starting with reaction (1), the transition can be characterized as

$$|W^+ \, p\rangle = |+1, +1/2\rangle \rightarrow |3/2, +3/2\rangle \rightarrow |\pi^+ p\rangle = |+1, +1/2\rangle \quad (7)$$

For every $\rightarrow$ in the above expression there is a C-G coefficient, which for this reaction are just "1" for both positions. The isopin I = 1 and I = 1/2 elements of the states are all completely aligned with the charge axis throughout the transition, and so the transition involves a single reduced amplitude, A_3, corresponding to the I = 3/2 states of the πN system. Thus the full set of coefficients for reaction (1) is

$$\sqrt{2} \cdot 1 \cdot A_3 \cdot 1 = \sqrt{2} \; A_3. \quad (8)$$

Reactions (2) and (3) can be decomposed in a similar way, however these latter reactions receive contributions from two reduced amplitudes, namely A_3 and A_1, the latter arising from the possible presence of I = 1/2 states of the πN system. For reaction (2) we have two possible (virtual) transition sequences:

$$|W^+ \, n\rangle = |+1, -1/2\rangle \rightarrow |3/2, +1/2\rangle \rightarrow |\pi^0 p\rangle = |0, +1/2\rangle \quad (9)$$

and

$$|W^+ \, n\rangle = |+1, -1/2\rangle \rightarrow |1/2, +1/2\rangle \rightarrow |\pi^0 p\rangle = |0, +1/2\rangle \, . \quad (10)$$

Together these yield the total transition amplitude

$$\sqrt{2} \left\{ \sqrt{\frac{1}{3}} \cdot A_3 \cdot \sqrt{\frac{2}{3}} + \sqrt{\frac{2}{3}} \cdot A_1 \cdot (-) \sqrt{\frac{1}{3}} \right\} = \frac{2}{3}(A_3 - A_1). \quad (11)$$

For reaction (3) we have

$$|W^+ \ n\rangle = |{+1}, -1/2\rangle \rightarrow |3/2, +1/2\rangle \rightarrow |\pi^+ n\rangle = |1, -1/2\rangle \tag{12}$$

and

$$|W^+ \ n\rangle = |{+1}, -1/2\rangle \rightarrow |1/2, +1/2\rangle \rightarrow |\pi^+ n\rangle = |1, -1/2\rangle \,. \tag{13}$$

From these we obtain the total transition amplitude

$$\sqrt{2}\left\{\sqrt{\frac{1}{3}} \cdot A_3 \cdot \sqrt{\frac{1}{3}} + \sqrt{\frac{2}{3}} \cdot A_1 \cdot \sqrt{\frac{2}{3}}\right\} = \frac{\sqrt{2}}{3}(A_3 + 2\,A_1). \tag{14}$$

In summary, we have obtained

$$\mathcal{A}\left(\nu_\mu \mathrm{p} \rightarrow \mu^- p \pi^+\right) = \sqrt{2}\ A_3, \tag{15}$$

$$\mathcal{A}\left(\nu_\mu \mathrm{n} \rightarrow \mu^- p \pi^0\right) = \frac{2}{3}(A_3 - A_1), \tag{16}$$

$$\mathcal{A}\left(\nu_\mu \mathrm{n} \rightarrow \mu^- n \pi^+\right) = \frac{\sqrt{2}}{3}(A_3 + 2\,A_1). \tag{17}$$

3 Relationships among amplitudes and cross sections

By multiplying Eq. (16) by $\sqrt{2}$ and adding it to Eq. (17) we may eliminate A_1 and relate the result to Eq. (15), obtaining

$$\mathcal{A}\left(\nu_\mu \mathrm{p} \rightarrow \mu^- p \pi^+\right) = \sqrt{2} \cdot \mathcal{A}\left(\nu_\mu \mathrm{n} \rightarrow \mu^- p \pi^0\right) + \mathcal{A}\left(\nu_\mu \mathrm{n} \rightarrow \mu^- n \pi^+\right) \tag{18}$$

From this we can develop the following triangular inequalities among the cross sections [3]:

$$\sqrt{2 \cdot \sigma\left(\nu_\mu \mathrm{n} \rightarrow \mu^- p \pi^0\right)} \ \leq \ \sqrt{\sigma\left(\nu_\mu \mathrm{p} \rightarrow \mu^- p \pi^+\right)} \ + \ \sqrt{\sigma\left(\nu_\mu \mathrm{n} \rightarrow \mu^- n \pi^+\right)} \tag{19}$$

and

$$\sqrt{\sigma\left(\nu_\mu \mathrm{n} \rightarrow \mu^- n \pi^+\right)} \ \leq \ \sqrt{\sigma\left(\nu_\mu \mathrm{p} \rightarrow \mu^- p \pi^+\right)} \ + \ \sqrt{2 \cdot \sigma\left(\nu_\mu \mathrm{n} \rightarrow \mu^- p \pi^0\right)}. \tag{20}$$

More generally, measurements of the three cross sections in an interval of neutrino energy can establish the average values of $|A_1|^2$, $|A_3|^2$, and $\mathrm{Re}\{A_1 A_3^*\}$ for that interval. This can be done as follows [3]:

$$\langle |A_3|^2 \rangle = \frac{1}{2}\sigma\left(\nu_\mu \mathrm{p} \rightarrow \mu^- p \pi^+\right), \tag{21}$$

$$\langle |A_1|^2 \rangle = \frac{3}{4}\left[\sigma\left(\nu_\mu \mathrm{n} \rightarrow \mu^- p \pi^0\right) \ + \ \sigma\left(\nu_\mu \mathrm{n} \rightarrow \mu^- n \pi^+\right) - \frac{1}{3}\sigma\left(\nu_\mu \mathrm{p} \rightarrow \mu^- p \pi^+\right)\right], \tag{22}$$

and

$$\langle \mathrm{Re}\{A_1 A_3^*\} \rangle = \frac{9}{32}\left[\sigma\left(\nu_\mu \mathrm{n} \rightarrow \mu^- n \pi^+\right) \ - \ 2 \cdot \sigma\left(\nu_\mu \mathrm{n} \rightarrow \mu^- p \pi^0\right) \ + \ \frac{1}{3}\sigma\left(\nu_\mu \mathrm{p} \rightarrow \mu^- p \pi^+\right)\right]. \tag{23}$$

3.1 Predictions in the case of I = 3/2 dominance

Of course a prominent feature of neutrino CC(π) scattering for E_ν of 1-2 GeV is the $\Delta(1232)$ resonance, which is an I = 3/2 particle. One might then consider that a useful approximation is to assume that $A_3 \gg A_1$. Equations (15), (16), and (17) then lead to the cross-section ratios

$$\sigma\left(\nu_\mu \mathrm{p} \to \mu^- p\pi^+\right) : \sigma\left(\nu_\mu \mathrm{n} \to \mu^- p\pi^0\right) : \sigma\left(\nu_\mu \mathrm{n} \to \mu^- n\pi^+\right) \;=\; 1 : \frac{2}{9} : \frac{1}{9}. \tag{24}$$

For scattering on a target that has equal numbers of protons and neutrons, the charge ratio for produced pions (assuming $A_3 \gg A_1$) is predicted to be

$$\frac{\sigma\left(\pi^+\right)}{\sigma\left(\pi^0\right)} = 5. \tag{25}$$

Equivalently, for $A_3 \gg A_1$, one may state the following ratios [1]:

$$R^+ = \sigma\left(\nu_\mu \mathrm{n} \to \mu^- n\pi^+\right) / \sigma\left(\nu_\mu \mathrm{n} \to \mu^- p\pi^0\right) \;=\; \frac{1}{2}, \tag{26}$$

$$R^{++} = \left[\sigma\left(\nu_\mu \mathrm{n} \to \mu^- p\pi^0\right) + \sigma\left(\nu_\mu \mathrm{n} \to \mu^- n\pi^+\right)\right] / \sigma\left(\nu_\mu \mathrm{p} \to \mu^- p\pi^+\right) \;=\; \frac{1}{3}, \tag{27}$$

$$R_1 = \sigma\left(\nu_\mu \mathrm{n} \to \mu^- p\pi^0\right) / \sigma\left(\nu_\mu \mathrm{p} \to \mu^- p\pi^+\right) \;=\; \frac{2}{9}, \tag{28}$$

$$R_2 = \sigma\left(\nu_\mu \mathrm{n} \to \mu^- n\pi^+\right) / \sigma\left(\nu_\mu \mathrm{p} \to \mu^- p\pi^+\right) \;=\; \frac{1}{9}. \tag{29}$$

3.2 Applications using bubble chamber measurements

In measurements conducted with the deuterium-filled ANL 12-ft diameter bubble chamber exposed to a wide band neutrino beam with spectral peak at $E_\nu \sim 05$ GeV [1], the above ratios were measured in samples with the nucleon-pion invariant mass restricted to less than 1.4 GeV [1]:

$$R^+ = 0.96 \pm 0.12, \quad R^{++} = 0.64 \pm 0.05, \quad R_1 = 0.33 \pm 0.04, \text{and } R_2 = 0.31 \pm 0.03. \tag{30}$$

Clearly these cross section ratios, measured near the threshold for CC(π) production, do not agree with the predictions from I = 3/2 dominance, hence it must be the case that the A_1 amplitude makes a sizable contribution – in addition to the A_3 amplitude. Using Eqs. (21), (22), and (23), the relative magnitudes of the amplitudes A_3 and A_1, together with their relative phase, were determined from the ANL data. For CC(π) production near threshold [1]:

$$|A_1|/|A_3| = 0.68 \pm 0.04 \text{ and } \phi_{13} = 90.7^\circ \pm 4.6^\circ. \tag{31}$$

The value of the relative phase indicates that the amplitudes A_1 and A_3 are almost 90° out-of-phase, on average. 'This can be qualitatively understood from the fact that the experiment spans the region around the (3, 3) resonance wherein the phase of A_3 varies from 0 to π (being $\frac{\pi}{2}$ at the resonance peak), while the phase of the nonresonant A_1 amplitude remains near 0.' [3].

4 Remarks on neutrino scattering in hydrocarbon

The cross sections for neutrino CC(π) samples reported by current generation experiments (as for example, Ref. [5]) necessarily use reaction channel definitions that are less “exclusive” than the reactions as listed in (1), (2), and (3). Nevertheless, the isospin decomposition that underwrites the relations presented here is of a general nature; it should be valid to good approximation for any sets of $CC(\pi)$ samples that share a similar definition of the hadronic sector of the interactions. Thus we expect the cross section relations of this Note to be relevant to CC(π) cross sections obtained by T2K, MINERvA and NOvA with the proviso that the channel definitions are done in similar ways and that they approximate the reactions as given by (1), (2), and (3).

References

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